The 3 Rs in new times: Research, rhetoric, and reality

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In this paper I argue that we need to analyse critically our currently popular theories of mathematical learning and instruction. I initially raise a number of issues warranting attention and begin by reviewing briefly a selection of theories, confining my discussion to socioconstructivist perspectives, situated cognition, and cognitive psychology /cognitive science. I then undertake a critical analysis of these theories, with a focus on the curriculum examples that have been used to support these theories. Finally, I propose a working model that might assist us in advancing mathematics education and research into new times.

Introduction: Imminent Upheaval

> Learning involves asking, investigating, formulating, representing, reasoning, and using strategies to solve problems, and then reflecting on the mathematics being used. (Romberg & Collins, in press)

Ideally, learning in the mathematics classroom would incorporate all of the above features. Reality, however, is another issue. I anticipate a major upheaval in the international mathematics education scene, ignited largely by concerns over students' performance on the recent Third International Mathematics and Science Study (TIMSS; National Research Council, 1996). Global reactions to these test results will have far-reaching implications for mathematics education research and curriculum development. Researchers need to be proactive, rather than reactive, in responding to this emerging upheaval in our field, especially since our work and our theories are receiving much of the blame for the TIMSS findings (McKeown, 1998).

I thus propose that we need to analyse critically our currently popular theories of mathematical learning and instruction. In particular, we need to address these questions:

- Are our current theories adequately informing us of students' learning in the mathematics classroom?
- How well coordinated are the theories and the learning experiences they recommend?
- Do these theories effectively support our development of rich curriculum experiences?

I would argue that it is difficult to respond positively to each of these questions and will attempt to demonstrate why in the course of this paper. My concerns regarding these issues are reflected in a forthcoming editorial by Judith Sowder (1998), in which she quotes a recent Californian project commissioned to synthesise K-12 Mathematics Education Research (Carnine, 1997). The guidelines for the project stipulated that a minimum identification requirement for a piece of research to be included in the analysis was that it be an experimental study. I will leave the reader to ponder the implications of this, especially in light of Carnine's (1997) comments on his study:

> The stakes surrounding this work are extremely high in California. Not only will the quality of mathematics education of hundreds of thousands of California students be affected by standards, curriculum frameworks, and instructional materials associated with this synthesis of experimental mathematics studies, but quite likely, the quality of mathematics education for millions of other students around the country and beyond (p. 2).

Theories Of Mathematics Learning And Instruction

Socioconstructivism

During the past decade or so, constructivist and sociocultural theories of learning have had a major impact on mathematics education research and curriculum reform (e.g., Cobb & Bauersfeld, 1995; Cobb & Yackel, 1996; Hickey, 1997; National Council of Teachers of Mathematics, 1989). Strong support for socioconstructivist approaches to mathematics education has come from studies documenting how students develop conceptions that deviate quite dramatically from those the teacher intended (Cobb & Bauersfeld, 1995).

It is not the intention here to review the various interpretations of socioconstructivism. The reader is referred to a recent article by Hickey (1997) for an overview of the different paradigms. Of relevance here is the predominant socioconstructivist view of mathematics education, one in which the "inquiry mathematics microculture" is at the core of students' learning and communication processes (Cobb, Yackel, McClain, & Whitenack, 1997).

The Inquiry Mathematics Microculture: At the heart of the socioconstructivist paradigm is the "inquiry mathematics microculture," defined as "a classroom culture in which explanations and justifications carry the significance of acting on mathematical objects." These explanations involve "specifying instructions for manipulating symbols that do not necessarily signify anything beyond themselves" (Cobb & Bauersfeld, 1995, pp. 295-296). (In other words, mathematical constructs are operated on as entities in their own right, having moved beyond the processes or operations with which they were associated).

From the socioconstructivist perspective, inquiry mathematics has two dimensions, one cognitive and the other, sociological. In cognitive terms, inquiry mathematics entails creating and operating on "experientially real mathematical objects" (i.e., object-like structural conceptions), while the sociological component involves "participating in the development of a taken-as-shared mathematical reality" (Cobb, 1995, p. 104; Cobb, Boufi, McClain, & Whitenack, 1997). Learning within this inquiry microculture involves a process of self-organization, which occurs as students participate in, and contribute to, the development of mathematical practices established in the classroom. The mathematical concepts that each student individually constructs are seen as relative to their participation in these practices as well as constrained by it. The individual does not just internalize the meanings that were constructed during the social activity, rather, she reflects on the meanings, "replicating at a personal level what has occurred and is occurring on the social level" (Prawat, 1996, p. 220). As they participate in their classroom activities, students are considered to undergo a process enculturation into the mathematical practices of the wider community.

Students' beliefs about their own role, the role of others, and the general nature of school mathematical activity are important social norms that support the inquiry microculture (Cobb & Bauersfeld, 1995; Cobb & Yackel, 1996; Richards, 1991). Equally important are the sociomathematical norms which facilitate students' mathematical constructions and activities. These norms include: (a) established argumentation in which teacher and students challenge explanations that merely describe symbol manipulation, (b) acceptable explanations which take significance of acting on taken-as-shared mathematical objects, and (c) teacher and students acting in a taken-as-shared mathematical reality, enriching that reality as they negotiate mathematical meanings (Cobb & Bauersfeld, 1995).

Situated cognition, addressed next, shares some characteristics of the socioconstructivist approach, but has some special features.

Situated Cognition

In simple terms, situated cognition argues that a good deal of what is learned is specific to the situation in which it is learned. An indication of the theory's popularity is evident in the recent publication, *Sitiuated cognition: Social, semiotic, and psychological perspectives* (Kirschner & Whitson, 1997). As the editors noted in their preface to this volume, the main impetus for situated-cognition theory has been a dissatisfaction with other paradigms for exploring learning and knowledge "as processes that occur in a local, subjective, and socially constructed world" (p. vii). Situated cognition, as we know it today, has arisen from a number of prominent studies, including those of Lave (e.g., 1988), Walkerdine (e.g., 1988), Davydov and his colleagues (e.g., Davydov & Radzikhovskii, 1985), and, of course, the classic works of Vygotsky (e.g., 1981).

On the other hand, as Bereiter (1997) has pointed out, the notion of situated learning was prominent in the first half of this century. Albeit, the concept was derived from a "psychology of situated rat behavior," but it nevertheless had the important tenet that "animals do not simply learn responses, they learn their environments" (Bereiter, 1997, p. 281). Rats will fathom a maze under carefully controlled conditions, learning a fixed route to reach a goal. However, they soon become lost when conditions are changed. On the other hand, rats will readily explore on their own (as we all know), and they soon work out an efficient way of getting from point A to point B. In other words, rats interact actively with their environment. As Bereiter further noted, we tend to forget that animal cognition is also situated, and, for humans, the situatedness of our cognition has a biological basis, despite strong cultural influences. What we need to do, according to Bereiter, is overcome the "situatedness of cognition" (Bereiter, 1997, p. 283).

Nevertheless, there have been numerous studies in the past decade that have lent support to the theory of situated cognition in students' mathematical learning (e.g., Carraher, Carraher, & Schleimann, 1985; Nunes, Schleimann, & Carraher, 1993; Saxe, 1991). It is beyond the scope of this paper to address these studies in detail. It will suffice to say that the work of situated-cognition theorists has included a focus on: (i) the problems associated with the transfer of "school mathematical knowledge" to the outside world, (ii) the influence of social and cultural practices in the classroom on students' mathematical learning, and (iii) the question of abstraction processes in mathematics instruction (Anderson, Reder, & Simon, 1996; Lerman, 1998). Probably the most frequently cited arguments arising from the situated-cognition perspective pertain to the mismatch between students' mathematical learning in typical school situations and their learning in the real world, such as the students' home environment or their place of part-time employment, where they need to apply mathematical knowledge. Lave's (1988) explanation for such a mismatch is that:

> . . . arithmetic practices are made to fit the activity at hand, and there are discontinuities between the techniques used to solve arithmetic problems in school-like situations and in the situations of shopping, selling produce, cooking, making and selling clothes, and assembling truckloads of dairy products. Place-holding algorithms do not transfer from school to everyday situations, on the whole. On the other hand, extraordinarily successful arithmetic activity takes place in these chore and job settings. (p. 149)

In concluding this section, I quote Lave's (1997) comment that mathematics education researchers adopting the situated-cognition perspective would likely describe conventional school mathematics learning:

> . . . as the all too mechanical transmission of a collection of facts to be learned by rote, a process devoid of creative contributions by the learner" (p. 17).

Cognitive Psychology

> Mathematics . . . is the study of the structures that we use to understand and reason about our experience -- structures that are inherent in our preconceptual bodily experience and that we make abstract via metaphor. (Lakoff, 1987, p. 355)

One of the challenges that face research and development in mathematics education is how students mentally structure their mathematical experiences, how they reason with these structures in learning and problem solving, and how they demonstrate to us that they do understand (Davis & Maher, 1997; English, 1997a; Hiebert, 1998). Such challenges are of particular interest to researchers who draw upon theories derived from cognitive psychology and, more recently, cognitive science.

In contrast to the socioconstructivist perspective, one of the core tasks of cognitive psychology is modelling the nature of hypothesized knowledge structures and cognitive processes underlying the learning and application of mathematics (e.g., Anderson, 1990; Greeno, 1989; Ohlsson & Rees, 1991; Resnick & Singer, 1993). Cognitive processes may be seen as actions on mental representations, these being internal mental structures that correspond to a segment of the world. Mental representations are often viewed in terms of networks of interrelated ideas, with the degree of understanding determined by the number and strength of the connections (English & Halford, 1995; Hiebert & Carpenter, 1992).

Cognitive researchers generally agree that we need to implement learning experiences that foster the construction of mental models or representations, which comprise the important relations and principles of a mathematical domain (e.g., Davis & Maher, 1997; English, in press a; Fuson, 1992; Goldin, 1992; Greeno, 1991; Hiebert, 1998; Romberg, 1998; Sfard, 1994). There has been considerable debate however, on how students construct these mental representations, on what forms these take and how they change with development, and on how students apply these representations in different mathematical situations.

Although each of the three main theoretical perspectives reviewed here recognize the importance of engaging children in mathematical problem solving, the cognitive psychologists have a particular interest in this issue. Some of our longstanding theories of problem solving were derived initially from the early information-processing models of human cognition, such as those of Newell and Simon (1972) and Simon (1978). Granted, though, these early models, which were tied closely to the computer metaphor, have some major limitations when applied to mathematical problem solving within and beyond the classroom (Greer, 1996).

Modern cognitive researchers generally agree that children learn mathematics best by solving problems, where they have the opportunity to work out the relationships between important ideas. Students develop understanding as they try to make sense of a problem, as they construct a mental model of it, and as they develop and refine methods of solution (Davis & Maher, 1997; English & Halford, 1995; Hiebert, 1998). As Hiebert (1998) emphasised, this understanding can be enhanced by engaging children in problems that are "nontrivial, multifaceted, and solvable using a variety of strategies" (p.142). Children's experiences with such problems should motivate them to "wonder why things are, to inquire, to search for solutions, and to resolve incongruities" (Hiebert, Carpenter, et al., 1996, p. 12).

In conjunction with this focus on problem solving has been an emphasis on the development of students' so-called "higher-order thinking skills" (e.g., Halpern, 1992; Paul, 1990; Peterson, 1988; Resnick & Resnick, 1992), and, more recently, on the development of their mathematical reasoning processes (e.g., analogical reasoning; English, in press b; English, 1997b). In recent times, cognitive psychology has been broadening into a cognitive science that draws upon several disciplines, including psychology, philosophy, computer science, linguistics, and anthropology (English, 1997a).

In the next section, I undertake a brief critical analysis of the key theories I have reviewed. In doing so, I highlight some of their perceived weaknesses, as indicated by others in the field. I follow this with a consideration of the curriculum activities used to support the theories.

Rhetoric and Reality: Time for Critical Analysis

> Research that aims to understand depends on the development of well-articulated theories . . . we need to think of theories not as grand global theories that unify the elements in mathematics education but rather as the products of making explicit our hypotheses and hunches about how things work . . . they [theories] provide explanatory solutions to the problems under study . . . it is well-recognized now that

all observations, all data, are theory-laden. We do not make neutral observations; our observations are biased by the theories we use (Hiebert, 1998, pp. 144-145)

As Hiebert (1998) noted, theories are useful if made explicit, even if they are incorrect or inadequate. Theories need to be open to scrutiny and revised on the basis of constructive feedback. However, there is the danger that incorrect theories can interfere with developments in understanding, if the theories are retained in the face of conflicting evidence; this applies to the activities of both children and researchers. The problem is, as Kuhn and her colleagues have shown repeatedly, we find it very difficult to revise or even discard our theories, even in the face of conflicting evidence (e.g., Kuhn, Schauble, & Garcia-Mila, 1992). We would far rather confirm our theories than disconfirm them.

So, what is the message here for our popular theories of mathematics learning and teaching? Clearly, we need to ensure that we explicate our theories as best we can, that we reflect critically on our theories and practices, and that we remain open to new evidence and new ideas.We need to actively search for these new ideas. I argue that we cannot rest on our existing theories and assume they will carry us into new times, because they won't - - not even with a delicate face lift. Now is the time for us to act. We need to begin by critically evaluating our theories, our research, our progress, and, most importantly, our impact within the mathematics classroom.

A critical assessment of our research activities is essential for deepening our understanding of mathematics learning. As Hiebert (1998) commented, mathematics education is "subject to wildly oscillating opinions about courses of action" (p. 148). Without fully understanding the situations we are researching, we are prone to simply following the fashion of the day and not making informed and rational decisions. We need to step back and ask ourselves <u>why</u> we are adopting a particular theory to guide our investigations. Is it because it is the "accepted" theory to adopt ("flavour of the decade," so to speak), or because our supervisors or colleagues use it (one wouldn't dare use anything else!), or because the theory is so well established that no one will question our use of it? If so, it is time to step back and rethink our mathematics education, its research, and its theories.

To set the critical ball rolling, I present below some quotes from researchers who have commented on the present theories.

> For social constructivists, the negotiation of meanings in social situations, perhaps taking place implicitly, on a meta-level, is as important as the individual constructions of the learner. In my view, the process of that learning is not clearly elaborated. (Lerman, 1998)

> The term, "construct," . . . is unnecessarily vague and misleading . . . Just as past curriculum reform failures occurred when "discovery" was treated as if it were an end in itself, regardless of the quality of what was discovered, curriculum reformers today often treat "construction" as an end in itself, regardless of what gets constructed. In both cases, the means to an end is is treated as the end in itself, while more important ends receive too little attention. (Lesh, Hoover, Hole, Kelly, & Post, in press)

> Without that core [the world of immaterial knowledge objects], formal education becomes meaningless (as, indeed, some advocates of situated cognition seem to believe it is). Again, for better or worse, formal education is our individual escape route from the confines of situated cognition. (Bereiter, 1997, pp. 283-284).

> Although situated cognition researchers have taken a lively interest in learning, both in and out of school, they have not come up with anything that could be called a new educational vision. Instead, situativity theorists have tended to endorse various innovations of a social constructivist cast, interpreting them within their own

> frameworks. . . The main difficulty, I would suggest, is that situativity theory has not been able to provide a cogent idea of the point of schooling. (Bereiter, 1997, p. 295)

> The educational implications of research into situated cognition are not altogether clear . . . the implications of situated cognition research for mathematics curricula, and for the teaching and learning of school mathematics, need to be investigated in creative ways. (Ellerton & Clements, 1998, p. 165)

> The classic information-processing conception was based largely on research on human learning conducted in artificial laboratory environments. . . What was needed was a shift to the study of human cognition in more realistic contexts. In a sense, by holding too tightly to a literal view of the human-computer analogy, information-processing psychology took the field on a 20-year detour around a central challenge in cognitive science-- the explanation of human cognition in realistic settings. (Mayer, 1996, p. 159).

I now address some of the curriculum examples that have been used to illustrate the theories. I leave it to the reader to undertake a more critical analysis of these examples, and simply raise a few questions for further consideration.

The Inquiry Mathematics Microculture

The following is an example from Yackel and Cobb (1996), which they provided to illustrate their theory, in particular, to demonstrate sociomathematical norms in action. This episode occurred in a classroom of young children. The number sentences, 16 + 14 + 8 = __ and 78 - 53 = __ , were written on the chalkboard and posed as mental computation activities. The children were encouraged to generate their own meaningful ways of solving these examples, and to justify and explain their solutions. An excerpt from the teacher's and children's interactions is given below:

> <u>Lemont:</u> I added the two 1s out of the 16 and [the 14] . . . would be 20 . . . plus 6 plus 4 would equal another 10, and that was 30 plus 8 left would be 38.
> <u>Teacher:</u> All right. Did anyone add a little *different*? Yes?
> <u>Ella:</u> I said 16 plus 14 would be 30 . . . and add 8 more would be 38.
> <u>Teacher :</u> Okay! Jose? *Different*?
> <u>Jose:</u> I took two tens from the 14 and the 16 and that would be 20 . . . and then I added the 6 and the 4 that would be 30 . . . then I added the 8, that would be 38.
> <u>Teacher:</u> Okay! It's almost similar to -- (Addressing another student) Yes? *Different*? All right. (Yackel & Cobb, 1996, pp. 462-463)

In analyzing these interactions, Yackel and Cobb (1996) highlighted the teacher's emphasis on the children finding *different* ways of solving the problems. This is seen as important in developing the classroom sociomathematical norms, as Yackel and Cobb explained:

> . . . in responding to the teacher's requests for different solutions, the students were simultaneously learning what counts as mathematically different and helping to constitute what counts mathematically different in their classroom. It is in this sense that we say the meaning of *mathematical difference* was interactively constituted by the teacher and the children. The teacher's responses and actions constrained the students' developing understanding of mathematical difference and the students' responses contributed to the teacher's developing understanding. (Yackel & Cobb, 1996, p. 462).

The authors also considered this classroom episode to be illustrative of how students develop particular mathematical beliefs and values, and consequently, how they develop intellectual autonomy in mathematics; in other words, how they develop a mathematical disposition.

The purpose of inquiry mathematics within the socioconstructivist paradigm is to foster students' ability to operate mentally on mathematical constructs as objects in their own right, that is, removed from the processes or operations with which the ideas were originally associated. In distinguishing between a traditional mathematics classroom and their inquiry microculture, Cobb and Bauersfeld (1995) referred to established practices of argumentation, where students and their teacher challenge explanations that merely describe symbol manipulation. The authors chose arithmetical examples from the early grades to illustrate these practices, primarily because the well-established, research-based models of children's arithmetical learning could guide the authors' instructional development. I raise some questions concerning this work:

1. What is really meant by the term, "inquiry," in describing the inquiry mathematics microculture?
2. Where is the evidence of students in this microculture developing a range of mathematical thinking, reasoning, and communication processes (e.g., thinking critically, creatively, and flexibly, posing mathematical questions and problems, and developing a commitment to the processes of mathematical inquiry and their improvement)?
3. What would be the key features of this microculture when other important mathematical domains are explored, such as geometry, probability, novel problem solving, and problem posing?
4. How would one describe an inquiry mathematics microculture operating at higher grade levels? [1]

Situated Cognition

In describing the learning of mathematics from a weight-watcher's perspective, Lave (1997) reported that the participants performed consistently better in solving isomorphic mathematics problems in other settings than in scholastic ones. In exploring their findings, Lave and her co-researchers tried to invoke the dieters' school mathematical knowledge during the course of their meal preparations. Given their failure to do so, Lave raised what she considered to be a central question: "Wherein lies the motivation for generating and solving problems in settings where there are not set tasks imposed on the problem solver?" (p. 24). I think the answer to this question is rather obvious. As part of her explanation, Lave explained that mathematical problem solving is not an end in itself for chefs, especially the weight-watching types. The mathematical relations they encounter when preparing their meals are important and meaningful to them in their efforts to reach a desired weight; that is, they have a sense of ownership over their problems. In dealing with the mathematical relations that arise, these weight-watching chefs would rarely employ algorithmic procedures taught in school. As Lave explained, "algorithmic math gets in the way" (Lave, 1997, p. 26). Instead, the chefs would invent a myriad of units of measurement and procedures for working out the required portions of food (e.g., using part of a design on a drinking glass to measure their milk allowance).

In support of her theory, Lave (1997) argued that mathematics education that focuses on detailed procedures for applying algorithmic knowledge to problems around the globe is not the answer:

> The problem is that any curriculum intended to be a specification *of* practice, rather than an arrangement of opportunities *for* practice . . . is bound to result in a teaching of misanalysis of practice . . . and the learning of still another. At best, it can only induce a new and exotic kind of practice contextually bound to the 'educational' setting. . . . In the settings for which it is intended (in everyday transactions), it will appear out of order and will not in fact reproduce good practice. (pp. 32-33).

[1] Cobb (in press) addresses, in part, some of the issues raised here.

As previously noted, the message from situated-cognition theory for mathematics education is not clear, at least for the Australian curriculum that we have achieved (that is, before the Benchmarks hit the scene). In addition to requesting that situated theorists become more familiar with contemporary mathematics education, I raise the following issues for consideration:

1. Is the theory of situated cognition one that can be reasonably adopted by mathematics educators? (see Sfard's, 1998)
2. If so, how might we apply the theory effectively to classroom practice?
3. Can theories of situated cognition help us to overcome the problems of transfer of learning? If so, how?

With respect to the above questions, it is worth noting the comments of Bereiter (1997) and Anderson, Reder, and Simon (1996). According to Bereiter, the main weakness of situated cognition is "precisely its situatedness" (p. 286). This poses transfer problems in both out-of-school and in-school learning situations. The proponents of situated cognition tend to highlight the latter. However, as Anderson et al. pointed out, the claims of these theorists "demonstrate at most that particular skills practiced in real-life situations do not generalize to school situations. They assuredly do not demonstrate the converse" (p. 6).

Cognitive Psychology

In a forthcoming article, Fischbein (in press) analyses the relationship between cognitive psychology, as a general theoretical framework, and "the psychology of mathematics education." He argues that the latter is not obtained by applying cognitive psychology to mathematics education, but, rather, by "identifying genuine psychological problems in mathematical activities." Fischbein elaborates by claiming that this can be achieved by "creating adequate specific concepts which psychology does not provide, by devising adequate specific research strategies, and by formulating safe interpretations that are meaningful for both cognitive psychology and the didactics of mathematics." I think these comments nicely illustrate the limitations in applying any one theoretical perspective to understanding and advancing students' mathematical learning. I elaborate on this point in the final section of this paper.

The other issue I wish to highlight with respect to the limitations of cognitive psychology is the narrow focus on mathematical problem solving adopted by many advocates of cognitive theories. These researchers rarely seem to move beyond the traditional computational problems (e.g., *Kerry-Ann has saved $29. Justin has saved $42. How much more has Justin saved than Kerry-Ann?;* e.g., Fennema, Carpenter, et al., 1996; Mayer & Hegarty, 1996; Verschaffel & De Corte, 1997). Albeit, the focus now is on exploring the invented strategies that children use in solving such problems. However, while cognisant of the continued difficulties these word problems present children, and the need for us to further our research in this area, it is of concern that cognitive studies are not branching out into other problems. These include novel problems that develop a broad range of reasoning processes, which are fundamental to students' mathematical development (e.g., Bransford et al., 1996; English, in press c). Indeed, there have been recent calls for mathematics curricula to place a greater focus on problems of this nature (Mathematical Association of America, 1998).

In proposing that cognitive research needs to broaden its perspectives on mathematics learning, I refer the reader to the work of Bransford et al. (1996). They initially investigated children's solutions to the typical word problems and simply reproduced the findings of numerous earlier studies. These included the finding that "mathematical thinking," for many children, is the procedures used for solving numerical problems (often a search for key words). As a consequence, Bransford et al. decided to explore other ways of designing problem experiences so that children would not only construct meaningful mental models, but also would develop a range of important concepts, reasoning, and communication processes. Bransford et al.'s development of specially designed video supports is well documented (e.g., Cognition and Technology Group at Vanderbilt, 1996). By immersing children in a diverse range of problem situations, they were able to move the children's problem-solving experiences (and their own research) beyond "well-defined word problems" to include problem generation, problem projects, and ultimately, "mathematics-driven adventures" (p. 211-216). Likewise, my own research has highlighted the importance of students creating their own problems, sharing their problems with peers, and providing each other with constructive feedback for problem improvement and extension (English, 1998a, 1998b, 1997 c, 1997d). These problem-posing activities warrant further attention from mathematics educators and researchers.

In the final section of this paper, I draw upon some of the issues I have addressed and outline some suggestions as to how we might advance mathematics education and research into new times.

Moving into New Times: What Now for Mathematics Education and Research?

> . . . in accounting for the development of mathematical cognition in a child, an analysis of a series of participations in social situations needs to be complemented by some account of the coherent development and restructuring of that individual's knowledge and conceptualizations over extended periods of time. (Greer, 1996, p. 185)

A suggested working model for addressing mathematics education and research appears in Figure 1. I use the term, "working," to indicate that the model is an evolving one, open to modification and refinement from ongoing theoretical and empirical analyses. In proposing this psychological and socio-philosophical model, my sentiments reflect those of Greer (1996), cited above, and below:

> Improving mathematics education is a massively complex human problem, in the cause of which all relevant forms of knowledge need to be mobilized. In pursuit of the (perhaps unattainable) goal of a comprehensive theory of learning mathematics, multiple contributions are required. (Greer, 1996, p. 191).

As can be seen in Figure 1, the model comprises multiple components, all of which I consider to be fundamental to the design and implementation of rich and meaningful learning experiences in mathematics. I argue that these elements must be addressed in conjunction, not in isolation, if we are to make the progress in mathematics education and research that is needed for the new millennium.

The Psychological and Socio-philosophical Components of Mathematics Education and Research

Psychological Components:

- Consideration of the nature and structure of mathematics;
- Critical analyses of theories of cognition and cognitive development whose primary focus is on the individual's mental models and processes (cf. Greer, 1996);
- Analyses of different forms of mathematical representations (concrete and abstract);
- A focus on different processes of mathematical reasoning and thinking;
- Analyses of the relational complexity of mathematical tasks (English, in press c);
- A focus on problem structures, problem finding, problem generation, and problem critiquing (English, Cudmore, & Tilley, in press; English, 1998b).

Socio-philosophical Components: Individual and Collective Mathematical Inquiry:

- Promoting connected dialogue on mathematical issues, including conceptual and affective issues (English & Cudmore, 1998);
- Fostering critical, constructive, and creative analyses of individual and shared ideas;
- Conducting philosophical debate on mathematical viewpoints, meanings, and dispositions;
- Establishing a community atmosphere of trust and respect for participants and their ideas.

I will not elaborate on all of the above, rather, I will simply address a few key points pertaining to the socio-philosophical components. My notion of mathematical inquiry extends beyond the socioconstructivist perspective of reasoning with computational objects; it permeates all areas of mathematical learning. The development of a shared spirit of mathematical inquiry hinges critically on the reconstruction of the entire classroom as an environment in which open questioning predominates, and where both students and their teachers take responsibility for the asking and answering of questions (cf. Splitter & Sharp, 1995). Members of the class build on, shape, and modify one another's ideas, offer and analyse reasons for arguments put forward, help one another formulate questions and pose mathematical problems, and clarify and justify their mathematical ideas. For such constructive communication to occur readily, students need the confidence in offering their ideas and points of view about mathematics, and in correcting their own reasoning and that of others, if such reasoning appears faulty. We need to engender that confidence in our students.

This model of an inquiring community builds on the links between modern philosophy and mathematics. Effective mathematics education utilises many of the strategies, processes, dispositions, and understandings that are directly addressed in philosophy (English, in preparation). Furthermore, philosophical inquiry provides criteria for examining effective thinking and reasoning, while, at the same time, strengthening the dispositions and skills associated with forming reasoned judgements and decisions (Splitter & Sharp, 1995). These common links between philosophy and mathematics have the potential to enhance not only students' mathematical development, but also, that of their teachers. For example, teachers who bring a philosophical dimension to their subject matter are likely to be more reflective, critical, and self-correcting in their instructional approaches; likewise they can transfer this disposition to the students within the classroom community (Splitter & Sharp, 1995). Such a critical and reflective approach is important in teachers' understanding of the conceptual structure of their discipline. Teaching which reflects this understanding is critical in mathematics education, especially as we face some turbulent times ahead.

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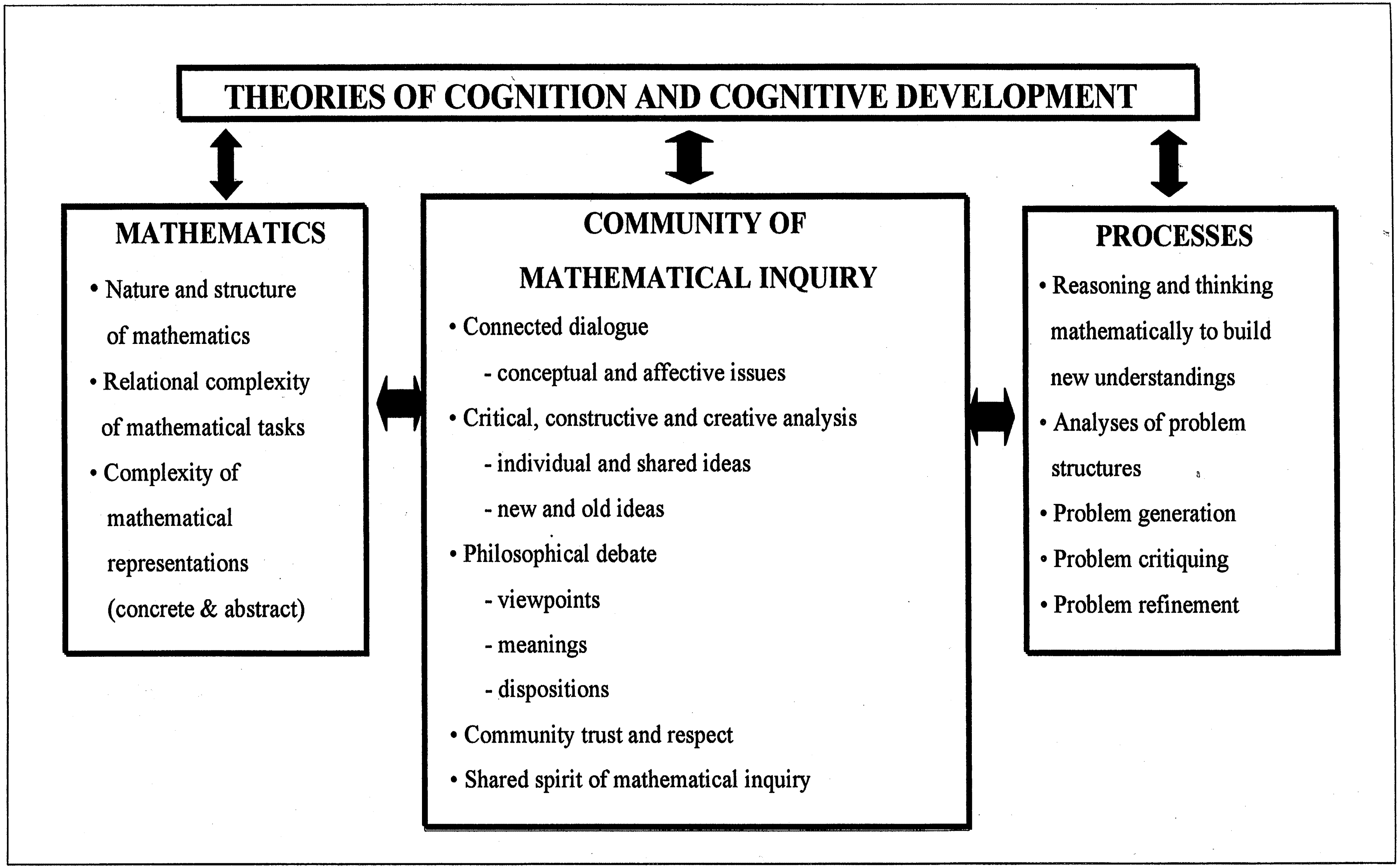

Figure 1. A Psychological and Socio-philosophical Working Model of Mathematics Education and Research.